14.8 **4. a.** Solve the system

$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

Soln: a) $\begin{cases} u = 2x - 3y & (1) \\ v = -x + y & (2) \end{cases}$ $(1) + 2(2)$: $u + 2v = \cancel{2x} - 3y \cancel{-2x} + 2y$

$= -y$.

$\Rightarrow y = -u - 2v.$

Substitute into (2):

$v = -x - u - 2v \Rightarrow x = -u - 3v$

So $\begin{cases} x = -u - 3v \\ y = -u - 2v \end{cases} \Rightarrow \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix}$

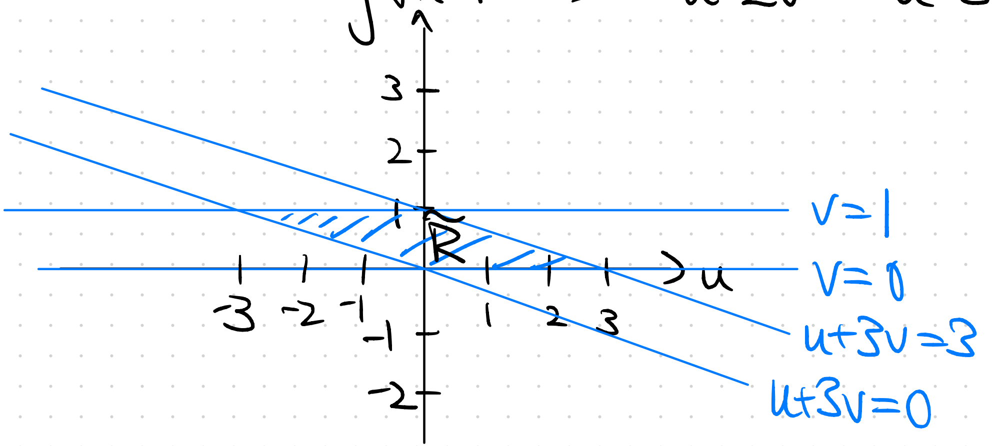
and $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix} = (-1)(-2)-(-3)(-1) = \boxed{-1}$

b) By part (a), $x=-3 \Rightarrow -3=-u-3v \Rightarrow u+3v=3$

$x=0 \Rightarrow 0=-u-3v \Rightarrow u+3v=0$

$y=x \Rightarrow -u-2v=-u-3v \Rightarrow v=0$

$y=x+1 \Rightarrow -u-2v=-u-3v+1 \Rightarrow v=1$



14.8 **8.** Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x-y)\,dx\,dy.$$

Sol'n: By part (a) of 14.8.4, $x-y = -u-3v-(-u-2v)$
$= -v.$

So $$\iint_R 2(x-y)\,dx\,dy = \iint_{\tilde{R}} 2(-v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^1 \int_{-3v}^{3-3v} -2v\,du\,dv = \int_0^1 -2vu\Big|_{u=-3v}^{u=3-3v} dv$$

$$= \int_0^1 \left(-6v+6v^2-6v^2\right) dv = \int_0^1 -6v\,dv = -3v^2\Big|_{v=0}^{v=1} = \boxed{-3}$$

14.8

12. The area of an ellipse The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk $G: u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

Sol'n: $\begin{cases} x = au \\ y = bv. \end{cases}$ So $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\frac{\partial(x,y)}{\partial(u,v)} = ab.$

Then $\iint dx\,dy = \iint_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_G ab\,du\,dv \underset{\substack{\uparrow \\ \text{Polar coordinates}}}{=} \int_0^{2\pi}\int_0^1 ab\,r\,dr\,d\theta$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$= \int_0^{2\pi} \frac{ab}{2} r^2 \Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{ab}{2} d\theta = \boxed{\pi ab}$

14.8

16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx.$$

(*Hint:* Show that the image of the triangular region G with vertices $(0,0)$, $(1,0)$, $(1,1)$ in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Sol'n:

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = (2u)(2u) - (-2v)(2v) = 4u^2 + 4v^2$$

Limits of integration are:

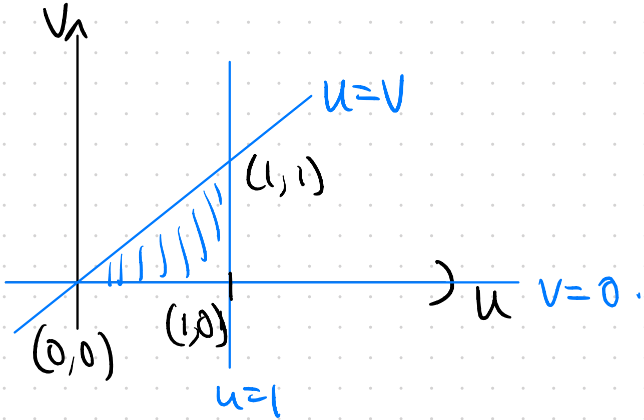
$x=0 \Rightarrow u^2 = v^2 \Rightarrow u = v$ or $u = -v$. (rej.).

$x=1 \Rightarrow 1 = u^2 - v^2$

$y=0 \Rightarrow 0 = 2uv \Rightarrow u=0$ or $v=0$

$y = 2\sqrt{1-x} \Rightarrow \cancel{2}uv = \cancel{2}\sqrt{1-u^2+v^2} \Rightarrow u^2v^2 = 1 - u^2 + v^2$

$\Rightarrow u^2(1+v^2) = 1+v^2 \Rightarrow u^2 = 1 \Rightarrow u = 1$ or -1 (rej.)



and $\sqrt{x^2+y^2} = \sqrt{(u^2-v^2)^2+(2uv)^2}$

$= \sqrt{u^4-2u^2v^2+v^4+4u^2v^2}$

$= \sqrt{u^4+2u^2v^2+v^4}$

$= \sqrt{(u^2+v^2)^2}$

$= u^2+v^2$

So applying the transformation, we have

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^1 \int_0^u (u^2+v^2)\left|4u^2+4v^2\right| dv\,du$$

$$= \int_0^1 \int_0^u (4u^4 + 8u^2v^2 + 4v^4)\,dv\,du$$

$$= \int_0^1 \left(4u^4 v \Big|_{v=0}^{v=u} + \frac{8}{3} u^2 v^3 \Big|_{v=0}^{v=u} + \frac{4}{5} v^5 \Big|_{v=0}^{v=u} \right) du$$

$$= \int_0^1 \left(4 + \frac{8}{3} + \frac{4}{5}\right) u^5\,du = \frac{112}{15} \int_0^1 u^5\,du = \frac{56}{45} u^6 \Big|_{u=0}^{u=1} = \boxed{\frac{56}{45}}$$

14.8

20. Let D be the solid region in xyz-space defined by the inequalities

$$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1.$$

Evaluate

$$\iiint\limits_D (x^2y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Sol'n: $x = u$, $y = \frac{v}{u}$, $z = \frac{w}{3}$ $\Rightarrow$ $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -vu^{-2} & u^{-1} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

and $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} 1 & 0 & 0 \\ -vu^{-2} & u^{-1} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \frac{1}{3u}$

$1 \le x \le 2 \Rightarrow 1 \le u \le 2$ $\qquad x^2 y = u^2 \frac{v}{u} = uv.$

$0 \le xy \le 2 \Rightarrow 0 \le v \le 2$ $\qquad 3xyz = \cancel{3}\cancel{u} \cdot \frac{v}{\cancel{u}} \cdot \frac{w}{\cancel{3}} = vw.$

$0 \le z \le 1 \Rightarrow 0 \le w \le 3$,

So applying transformation, we have

$$\iiint_D (x^2y + 3xyz)\,dx\,dy\,dz = \int_0^3 \int_0^2 \int_1^2 (uv + vw)\left|\frac{1}{3u}\right| du\,dv\,dw$$

$$= \int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\,dv\,dw = \int_0^3 \int_0^2 \left(\frac{vu}{3}\Big|_{u=1}^{u=2} + \frac{1}{3} vw \ln u \Big|_{u=1}^{u=2}\right) dv\,dw$$

$$= \int_0^3 \int_6^2 \left(\frac{1}{3}v + \frac{1}{3} vw \ln(2)\right) dv\,dw = \frac{1}{3}\int_0^3 \left(\frac{1}{2}v^2\Big|_{v=0}^{v=2} + \frac{1}{2}v^2 w \ln(2)\Big|_{v=0}^{v=2}\right) dw$$

$$= \frac{1}{3}\int_0^3 \left(2 + 2w\ln(2)\right) dw = \frac{2}{3}\left(w\Big|_{w=0}^{w=3} + \frac{1}{2}w^2\ln(2)\Big|_{w=0}^{w=3} \right)$$

$$= \boxed{2 + 3\ln(2)}$$

14.8

22. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a. $x = u\cos v, \quad y = u\sin v, \quad z = w$

b. $x = 2u - 1, \quad y = 3v - 4, \quad z = (1/2)(w - 4)$.

Sol'n: a) $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \cos v \det \begin{bmatrix} u\cos v & 0 \\ 0 & 1 \end{bmatrix} - (-u\sin v)\det \begin{bmatrix} \sin v & 0 \\ 0 & 1 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} \sin v & u\cos v \\ 0 & 0 \end{bmatrix}$

$$= u\cos^2 v + u\sin^2 v = \boxed{u}$$

b) $$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

and $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = \boxed{3}$$

13.1

10. Evaluate $\int_C (x - y + z - 2)\, ds$, where C is the straight-line segment $x = t$, $y = (1 - t)$, $z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.

Sol'n: Parameterize C by

$$r(t) = (x(t), y(t), z(t)) = (t, 1-t, 1) \text{ for } t \in [0, 1].$$

then $r'(t) = (1, -1, 0)$ and $\|r'(t)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

$$f(r(t)) = f(t, 1-t, 1) = t - (1-t) + 1 - 2$$

$$= 2t - 2.$$

So $$\int_C (x - y + z - 2)\, ds = \int_0^1 f(r(t))\, \|r'(t)\|\, dt$$

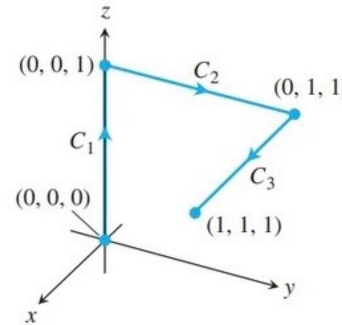
$$= \int_0^1 (2t-2)\sqrt{2}\, dt = 2\sqrt{2}\left(\tfrac{1}{2}t^2 - t\right)\Big|_0^1 = \sqrt{2} - 2\sqrt{2}$$

$$= \boxed{-\sqrt{2}}$$

15.1

16. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path C_1 followed by C_2 followed by C_3 from $(0, 0, 0)$ to $(1, 1, 1)$ (see accompanying figure) given by

$$C_1:\quad \mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1$$
$$C_2:\quad \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$
$$C_3:\quad \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1.$$



Sol'n: $r_1(t) = (0, 0, t)$, $r_1'(t) = (0, 0, 1)$ and $\|r_1'(t)\| = 1$

$r_2(t) = (0, t, 1)$, $r_2'(t) = (0, 1, 0)$ and $\|r_2'(t)\| = 1$

$r_3(t) = (t, 1, 1)$, $r_3'(t) = (1, 0, 0)$ and $\|r_3'(t)\| = 1$.

$f(r_1(t)) = f(0, 0, t) = -t^2$

$f(r_2(t)) = f(0, t, 1) = \sqrt{t} - 1^2 = \sqrt{t} - 1$

$f(r_3(t)) = f(t, 1, 1) = t + \sqrt{1} - 1 = t.$

So $I = \int_{C_1} (x+\sqrt{y}-z^2)\,ds + \int_{C_2} (x+\sqrt{y}-z^2)\,ds + \int_{C_3} (x+\sqrt{y}-z^2)\,ds$

$$= \int_0^1 -t^2\,dt + \int_0^1 (\sqrt{t}-1)\,dt + \int_0^1 t\,dt$$

$$= -\frac{1}{3}t^3\Big|_0^1 + \frac{2t^{3/2}}{3}\Big|_0^1 - t\Big|_0^1 + \frac{1}{2}t^2\Big|_0^1$$

$$= -\frac{1}{3} + \frac{2}{3} - 1 + \frac{1}{2} = \boxed{\frac{-1}{6}}$$

15.1

22. Find the line integral of $f(x, y) = x - y + 3$ along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

Sol'n: $r(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi.$

$r'(t) = (-\sin t, \cos t)$

$\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$

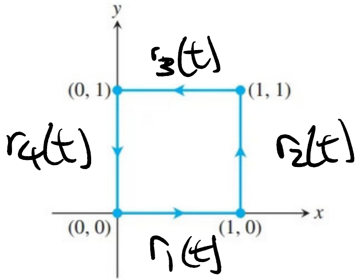
$f(r(t)) = \cos t - \sin t + 3.$

So $I = \int_0^{2\pi} (\cos t - \sin t + 3)\, dt = \sin t \Big|_0^{2\pi} + \cos t \Big|_0^{2\pi} + 3t \Big|_0^{2\pi}$

$= \boxed{6\pi}$

15.1

26. Evaluate $\int_C \frac{1}{x^2+y^2+1}\,ds$, where C is given in the accompanying figure.



Sol$_n$: $r_1(t) = (t, 0)$, $0 \le t \le 1$, $r_1'(t) = (1, 0)$, and $\|r_1'(t)\| = 1$

$r_2(t) = (1, t)$, $0 \le t \le 1$, $r_2'(t) = (0, 1)$ and $\|r_2'(t)\| = 1$

$r_3(t) = (1-t, 1)$, $0 \le t \le 1$, $r_3'(t) = (-1, 0)$ and $\|r_3'(t)\| = 1$

$r_4(t) = (0, 1-t)$, $0 \le t \le 1$, $r_4'(t) = (0, -1)$ and $\|r_4'(t)\| = 1$.

$f(x,y) = \frac{1}{x^2+y^2+1}$. So $f(r_1(t)) = \frac{1}{t^2+1}$, $f(r_2(t)) = \frac{1}{2+t^2}$

$f(r_3(t)) = \frac{1}{(1-t)^2+2}$, $f(r_4(t)) = \frac{1}{(1-t)^2+1}$

So $\int_C \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{t^2+1}dt + \int_0^1 \frac{1}{2+t^2}\,dt + \int_0^1 \frac{1}{(1-t)^2+2}\,dt + \int_0^1 \frac{1}{(1-t)^2+1}\,dt$

$$\int_0^1 \frac{1}{t^2+1}\,dt = \arctan(t)\Big|_{t=0}^{t=1} = \frac{\pi}{4}.$$

$$\int_0^1 \frac{1}{2+t^2}\,dt = \int_0^1 \frac{1}{2(1+\frac{t^2}{2})}\,dt = \frac{1}{2}\int_0^1 \frac{1}{1+\frac{t^2}{2}}\,dt = \frac{\sqrt{2}}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1+u^2}\,du$$

$u = \frac{t}{\sqrt{2}}$. $du = \frac{1}{\sqrt{2}}dt$. $\quad = \frac{\sqrt{2}}{2}\arctan(u)\Big|_{u=0}^{u=\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\right).$

$t=0 \Rightarrow u=0$

$t=1 \Rightarrow u=\frac{1}{\sqrt{2}}$

$$\int_0^1 \frac{1}{(1-t)^2+2}\,dt = \frac{1}{2}\int_0^1 \frac{1}{\frac{(1-t)^2}{2}+1}\,dt = -\frac{\sqrt{2}}{2}\int_{\frac{1}{\sqrt{2}}}^0 \frac{1}{u^2+1}\,du = -\frac{\sqrt{2}}{2}\arctan(u)\Big|_{u=\frac{1}{\sqrt{2}}}^{u=0}$$

$$= \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\right).$$

$u = \frac{1-t}{\sqrt{2}}$. $du = -\frac{1}{\sqrt{2}}dt$

$t=0 \Rightarrow u = \frac{1}{\sqrt{2}}$

$t=1 \Rightarrow u=0$

$$\int_0^1 \frac{1}{(1-t)^2+1}\,dt = -\int_1^0 \frac{1}{u^2+1}\,du = -\arctan(u)\Big|_{u=1}^{u=0} = \arctan\left(\frac{\pi}{4}\right).$$

$u = 1-t$ $du = -dt$

$t=0 \Rightarrow u=1$

$t=1 \Rightarrow u=0$

So adding all together, we have $\int_C \frac{1}{x^2+y^2+1}\,ds = \boxed{\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{2}}$